Trigonometric Identities:

Key Points:

- **Fundamental Identities:**

Pythagorean identities	$\cos^2\theta + \sin^2\theta = 1$ $1 + \cot^2\theta = \csc^2\theta$ $1 + \tan^2\theta = \sec^2\theta$
Even-odd identities	$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$ $\sin(-\theta) = -\sin\theta$ $\csc(-\theta) = -\csc\theta$ $\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$
Reciprocal identities	$\sin\theta = \frac{1}{\csc\theta}$ $\cos\theta = \frac{1}{\sec\theta}$ $\tan\theta = \frac{1}{\cot\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$
Quotient identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$
Cofunction identities	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$ $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

- **Sum and Difference Identities**

Sum Formula for Cosine	$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
Difference Formula for Cosine	$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
Sum Formula for Sine	$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
Difference Formula for Sine	$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
Sum Formula for Tangent	$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
Difference Formula for Tangent	$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

- **Double-Angle, Half-Angle, and Reduction Formulas:**

Double-angle formulas	$\sin(2\theta) = 2\sin\theta\cos\theta$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $= 1 - 2\sin^2\theta$ $= 2\cos^2\theta - 1$ $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$
Reduction formulas	$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ $\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Half-angle formulas	$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$ $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$ $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$ $= \frac{\sin\alpha}{1+\cos\alpha}$ $= \frac{1-\cos\alpha}{\sin\alpha}$

- **Sum-to-Product and Product-to-Sum Formulas**

Product-to-sum Formulas	$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$ $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$ $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$ $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$
Sum-to-product Formulas	$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$ $\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

- There are multiple ways to represent a trigonometric expression. Verifying the identities illustrates how expressions can be rewritten to simplify a problem.
- Simplifying one side of the equation to equal the other side is another method for verifying an identity.
- It is often useful to begin on the more complex side of the equation
- Verifying an identity may involve algebra with the fundamental identities.

Trigonometric Identities Videos:

- Verifying the Fundamental Trigonometric Identity: Examples 1-2
- Verifying the Fundamental Trigonometric Identity: Examples 3-4
- Using the Sum and Difference Formulas: Examples 5-6
- Using the Sum and Difference Formulas: Examples 7-8
- Using the Double Angle Formulas: Examples 9-11
- Using the Reduction Formulas: Example 12
- Using the Half Angle Formulas: Example 13
- Expressing Products as Sum: Examples 14-16
- Expressing Sum as Product: Example 17
- Verifying an Identity: Example 18

Practice Exercises:

1. **Use basic identities to simplify the expression.**

$$\sec x \ \cos x + \cos x - \frac{1}{\sec x}$$

2. **Determine if the given identities are equivalent.**

$$\sin^2 x + \sec^2 x - 1 = \frac{(1-\cos^2 x)(1+\cos^2 x)}{\cos^2 x}$$

For exercises 3 -7 use the sum and difference identities:

3. Find the exact value of: $\tan\left(\frac{7\pi}{12}\right)$

4. Find the exact value of : $\sin(70^{\circ})\cos(25^{\circ}) - \cos(70^{\circ})\sin(25^{\circ})$

5. Prove the identity: $\cos(4x) - \cos(3x)\cos(x) = \sin^2 x - 4\cos^2 x \sin^2 x$

6. Simplify the expression: $\frac{\tan\left(\frac{1}{2}x\right)+\tan\left(\frac{1}{8}x\right)}{1-\tan\left(\frac{1}{2}x\right)\tan\left(\frac{1}{8}x\right)}$

7. Find the exact value of **:** $\tan\left(\sin^{-1}(0) + \sin^{-1}\left(\frac{1}{2}\right)\right)$

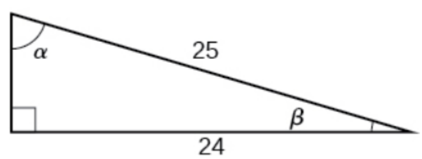
For exercises 8 – 12, use the Double-Angle, Half-Angle, and Reduction Formulas:

8. Find the exact value of **:** $\sin(2\theta)$, $\cos(2\theta)$ **and** $\tan(2\theta)$ given $\sec(\theta) = -\frac{5}{3}$ and θ is in the interval $[0, 2\pi]$

9. Find the exact value of $\sec\left(\frac{3\pi}{8}\right)$

10. Use the following figure to find the value of :

$\sin\left(\frac{\beta}{2}\right)$, $\cos\left(\frac{\beta}{2}\right)$, $\tan\left(\frac{\beta}{2}\right)$, $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, $\tan\left(\frac{\alpha}{2}\right)$



11. Verify the identity: $\cot(x)\ \cos(2x) = -\sin(2x) + \cot(x)$

12. Rewrite the expression with no powers: $\tan^2 x\ \sin^3 x$

For exercises 13 –16 , use Sum-to-Product and Product-to-Sum Formulas:

13. Find the exact value of: $2\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{5\pi}{6}\right)$

14. Find the exact value of: $\sin\left(\frac{\pi}{12}\right) - \sin\left(\frac{7\pi}{12}\right)$

15. Change the function from the product to a sum: $\sin(9x)\cos(3x)$

16. Change the function from sum to a product: $\sin(11x) + \sin(2x)$

Answers:

1. 1

2. Yes

3. $-2-\sqrt{3}$

4. $\frac{\sqrt{2}}{2}$

5.

$$
\begin{aligned}
\cos(4x)-\cos(3x)\cos x &= \cos(2x+2x)-\cos(x+2x)\cos\ x \\
&= \cos(2x)\cos(2x)-\sin(2x)\sin(2x)-\cos\ x\ \cos(2x)\cos\ x+\sin\ x\ \sin(2x)\cos\ x \\
&= \left(\cos^2 x-\sin^2 x\right)^2-4\cos^2 x\sin^2 x-\cos^2 x\left(\cos^2 x-\sin^2 x\right)+\sin\ x(2)\sin\ x\ \cos\ x\ \cos\ x \\
&= \left(\cos^2 x-\sin^2 x\right)^2-4\cos^2 x\sin^2 x-\cos^2 x\left(\cos^2 x-\sin^2 x\right)+2\sin^2 x\cos^2 x \\
&= \cos^4 x-2\cos^2 x\sin^2 x+\sin^4 x-4\cos^2 x\sin^2 x-\cos^4 x+\cos^2 x\sin^2 x+2\sin^2 x\cos^2 x \\
&= \sin^4 x-4\cos^2 x\sin^2 x+\cos^2 x\sin^2 x \\
&= \sin^2 x\left(\sin^2 x+\cos^2 x\right)-4\cos^2 x\sin^2 x \\
&= \sin^2 x-4\cos^2 x\sin^2 x
\end{aligned}
$$

6. $\tan\left(\frac{5}{8}x\right)$

7. $\frac{\sqrt{3}}{3}$

8. $-\frac{24}{25},-\frac{7}{25},\frac{24}{7}$

9. $\sqrt{2\left(2+\sqrt{2}\right)}$

10. $\frac{\sqrt{2}}{10},\frac{7\sqrt{2}}{10},\frac{1}{7},\frac{3}{5},\frac{4}{5},\frac{3}{4}$

11.

$$
\begin{aligned}
\cot x\cos(2x) &= \cot x\left(1-2\sin^2 x\right) \\
&= \cot x-\frac{\cos x}{\sin x}(2)\sin^2 x \\
&= -2\sin x\cos x+\cot x \\
&= -\sin(2x)+\cot x
\end{aligned}
$$

12.

$$\frac{10\sin x - 5\sin(3x) + \sin(5x)}{8(\cos(2x)+1)}$$

13. $\frac{\sqrt{3}}{2}$

14. $-\frac{\sqrt{2}}{2}$

15. $\frac{1}{2}(\sin(6x) + \sin(12x))$

16. $2\sin\left(\frac{13}{2}x\right)\cos\left(\frac{9}{2}x\right)$